

Localization and Rational Homotopy Type

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Outline

1 Introduction

2 Preliminaries

3 Localization

- Algebraic localization
- Localization of a space
- Cellular localization

Introduction

Rational Homotopy Theory begins with the discovery by Sullivan in the 1960's that simply connected topological spaces, X , and continuous maps between them, $f : X \rightarrow Y$, can themselves be rationalized

$$X \longrightarrow X_{\mathbb{Q}}$$

$$f_{\mathbb{Q}} : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$$

such that

$$H_*(X_{\mathbb{Q}}) = H_*(X; \mathbb{Q})$$

$$\pi_*(X_{\mathbb{Q}}) = \pi_*(X) \otimes \mathbb{Q}$$

Introduction

The rational homotopy type of a CW complex X is the homotopy type of $X_{\mathbb{Q}}$, and the rational homotopy class of $f : X \rightarrow Y$ is the homotopy class of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$.

Rational homotopy theory is then the study of properties that depend only on the rational homotopy type of a space or the rational homotopy class of a map.

The lost information in Rational Homotopy. The case of $\mathbb{S}^n$

	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
$\mathbb{S}^4$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_5$	$\mathbb{Z}_2$
$\mathbb{S}^5$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
$\mathbb{S}^6$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 1: A subset of the homotopy groups of the spheres $\mathbb{S}^4, \mathbb{S}^5$ and $\mathbb{S}^6$.

Something we know:

- Homotopy groups of $\mathbb{S}^k$ vanish until degree $k - 1$ (included).
- $\pi_k(\mathbb{S}^k) = \mathbb{Z}$
- π_n is commutative for $n \geq 2$
- $\pi_{2n}(\mathbb{S}^{2n}) \otimes \mathbb{Q} = \pi_{4n-1}(\mathbb{S}^{2n}) \otimes \mathbb{Q} = \mathbb{Q}$

Despite the lost information, rational homotopy has the advantage of being remarkably computational.

Due to Sullivan, it is known how to obtain the rational homotopy type of a topological space X , via the construction of a commutative differential graded algebra, called the Sullivan model of X . This algebra allows us to establish a categorical equivalence between rational homotopy types of spaces and isomorphism classes of Sullivan models:

$$\left\{ \begin{array}{l} \textit{Rational homotopy} \\ \textit{types of spaces} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \textit{isomorphism classes of} \\ \textit{minimal Sullivan models} \end{array} \right\}$$

Introduction

In this talk we are going to introduce the notion of localization of a space, and we will see the rationalization as a particular case. After that, we will define the rational homotopy type of a simply-connected space and the rational homotopy class of a map between such spaces.

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Preliminaries

For $n \geq 2$ the homotopy groups $\pi_n(X)$ are abelian.

Definition

Let X be simply-connected. The groups $\pi_n(X, x_0) \otimes \mathbb{Q}$ (for $n \geq 2$) are called the rational homotopy groups of (X, x_0) .

Preliminaries

Weak homotopy equivalence

A continuous map $f : (X, x_0) \rightarrow (Y, y_0)$ between pointed topological spaces is called a weak homotopy equivalence if the induced map

$$\pi_*(f) : \pi_*(X, x_0) \rightarrow \pi_*(Y, y_0)$$

is an isomorphism. The spaces X and Y have the same weak homotopy type if there is a finite chain of weak homotopy equivalences

$$X \leftarrow Z_1 \rightarrow \cdots \leftarrow Z_n \rightarrow Y$$

Preliminaries

Whitehead's Theorem

If a map $f : X \rightarrow Y$ between connected CW complexes induces isomorphisms $\pi_*(f) : \pi_*(X, x_0) \rightarrow \pi_*(Y, y_0), \forall n$, then f is a homotopy equivalence.

Whitehead-Serre

Let $f : X \rightarrow Y$ be a map between simply-connected spaces. Then the following assertions are equivalent for a subring R of $\mathbb{Q}$

- $\pi_*(f) \otimes_{\mathbb{Z}} R$ is an isomorphism
- $H_*(f, R)$ is an isomorphism

Preliminaries

CW-model

A cellular model or CW-model for a space Y is a CW-complex X together with a weak homotopy equivalence $f : X \rightarrow Y$

Existence and Uniqueness of CW-models

Every space Y has a CW-model $f : X \rightarrow Y$. This is unique up to homotopy equivalence.

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Algebraic localization

Assume all rings are integral domains with unit.

Let S be a multiplicative closed subset of a ring A . We define an equivalence relation $\sim$ on $A \times S$ via

$$(a, s) \sim (b, t) \iff (at - bs)u = 0$$

for some $u \in S$.

Localization of a commutative ring with unit

Denote by $S^{-1}A$ the set of all equivalence classes $a/s = [(a, s)]$ under the equivalence relation mentioned above.

$S^{-1}A$ has a well defined ring structure given by

$$(a/s) + (b/t) = (at + bs)/st$$

$$(a/s)(b/t) = ab/st$$

These operations give $S^{-1}A$ the structure of a commutative ring with unit.

Localization of a commutative ring with unit

There is a ring homomorphism:

$$A \rightarrow S^{-1}A$$

$$a \rightarrow a/1$$

Definition

The ring $S^{-1}A$ is called the localization of A in S .

Examples

- If $\mathbb{P}$ is a set of primes and S is multiplicative set of integers relatively prime to the elements of $\mathbb{P}$, then $S^{-1}\mathbb{Z}$ is the ring localized at $\mathbb{P}$, $\mathbb{Z}_{\mathbb{P}}$. It is a subring of $\mathbb{Q}$.
- The localized ring $\mathbb{Z}_{(5)}$ is a subring of $\mathbb{Q}$ where the fractions have denominators coprime to 5.
- $\mathbb{Z}_{\emptyset} = \mathbb{Q}$.
- $\mathbb{Z}_{all\ primes} = \mathbb{Z}$.

Localization of modules

In the same way, we can localize an R -module in a multiplicative set S

$$S^{-1}R \otimes_R M \cong S^{-1}M$$

In this case, we have an isomorphism of $S^{-1}R$ -modules due to $\gamma((a/s) \otimes m) = am/s$ with inverse $\gamma^{-1}(m/s) = 1/s \otimes m$.

S-local Module

Let M be an R -module. We call M an S -local module over a multiplicative set S of R , if $S^{-1}M \cong M$.

Notice that when $S^{-1}R = \mathbb{Q}$, $S^{-1}M$ becomes a $\mathbb{Q}$ -vector space. When $S = \mathbb{Z} \setminus \{0\}$, localization is called rationalization.

Localization of homotopy groups

The rational homotopy groups are the localization of the homotopy groups in $S = \mathbb{Z} \setminus \{0\}$:

$$(\pi_*(X, x_0))_{(0)} \cong \pi_*(X, x_0) \otimes \mathbb{Q}$$

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Localization of spaces

local/rational space, localization/rationalization

A simply-connected space X is called S -local if $\pi_*(X)$ is an S -local $\mathbb{Z}$ -module. An S -localization of a simply-connected space X is a map $\phi : X \rightarrow S^{-1}X$ to a simply-connected S -local space $S^{-1}X$ such that ϕ induces an isomorphism

$$\pi_*(X) \otimes S^{-1}\mathbb{Z} = S^{-1}\pi_*(X) \xrightarrow{\cong} \pi_*(S^{-1}X)$$

If $S = \mathbb{Z} \setminus \{0\}$ we say a rational space and rationalization. We denote the rationalization of the space X by $X_{\mathbb{Q}}$ (or $X_{(0)}$).

Localization of spaces

Localization (necessary and sufficient conditions)

Let Y be a simply-connected space. Y is S -local if and only if $H_*(Y, y_0; \mathbb{Z})$ is S -local. A continuous map $\phi : X \rightarrow Y$ between simply-connected spaces is an S -localization if and only if Y is S -local and $H_*(\phi; S^{-1}\mathbb{Z})$ is an isomorphism.

Localization of spaces

Existence

For each simply-connected space X there is a relative CW-complex $(S^{-1}X, X)$ (with neither 0-cells nor 1-cells) such that the inclusion $\phi : X \hookrightarrow S^{-1}X$ is an S -localization.

We call such a relative CW-complex $(S^{-1}X, X)$ a cellular localization of X .

Localization of spaces

Universal property

Let $(S^{-1}X, X)$ be a cellular localization. Suppose $f : X \rightarrow Y$ is a continuous map to a simply-connected S -local space Y . Then, f extends over $S^{-1}X$ to a map $\tilde{f} : S^{-1}X \rightarrow Y$. This map is unique up to homotopy, i.e., any two extensions of f are homotopic relative to X .

Properties of an S -localization of a space

- An S -localization is unique up to weak homotopy type
- Any two cellular S -localizations $(S^{-1}X, X)$ and $((S^{-1}X)', X)$ are homotopy equivalent.
- Let $S^{-1}X$ and $S^{-1}Y$ be cellular localizations. A map $f : X \rightarrow Y$ of simply-connected spaces induces a map $S^{-1}f : S^{-1}X \rightarrow S^{-1}Y$ such that the localization diagram commutes. The map $S^{-1}f$ is unique up to homotopy relative to X .

Rational homotopy equivalence

Definition

Let $f : X \rightarrow Y$ be a map between simply-connected topological spaces. Then f is called a rational homotopy equivalence if one of the following equivalent conditions is satisfied:

- $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism
- $H_*(f; \mathbb{Q})$ is an isomorphism
- $H^*(f; \mathbb{Z}) \otimes \mathbb{Q}$ is an isomorphism
- The rationalized map $f_{\mathbb{Q}} : X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ is a weak homotopy equivalence.

Rational Homotopy Type

Definition

The rational homotopy type of a simply-connected topological space X is defined to be the weak homotopy type of its rationalization $X_{\mathbb{Q}}$.

By the Whitehead-Serre theorem, we know that rational homotopy groups and rational homology groups are invariants of the rational homotopy type.

Whitehead-Serre

Let $f : X \rightarrow Y$ be a map between simply-connected spaces. Then the following assertions are equivalent for a subring R of $\mathbb{Q}$

- $\pi_*(f) \otimes_{\mathbb{Z}} R$ is an isomorphism
- $H_*(f, R)$ is an isomorphism

Rational Homotopy Type

Whitehead's Theorem

If a map $f : X \rightarrow Y$ between connected CW complexes induces isomorphisms $\pi_*(f) : \pi_*(X, x_0) \rightarrow \pi_*(Y, y_0), \forall n$, then f is a homotopy equivalence.

Moreover, if both X and Y are CW-complexes, so are their (cellular) rationalizations. Due to Whitehead's theorem, we know that the weak homotopy equivalence is a homotopy equivalence in the category of CW-complexes.

For X, Y CW-complexes, we can say that X and Y have the same rational homotopy type if and only if there is a homotopy equivalence between their rationalizations:

$$X_{\mathbb{Q}} \xrightarrow{\cong} Y_{\mathbb{Q}}$$

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Cellular localization

Now we are going to see (briefly) how can we localize a space via cellular localization.

Existence

For each simply-connected space X there is a relative CW-complex $(S^{-1}X, X)$ (with neither 0-cells nor 1-cells) such that the inclusion $\phi : X \hookrightarrow S^{-1}X$ is an S -localization.

We call such a relative CW-complex $(S^{-1}X, X)$ a **cellular localization of X** .

For that, we are going to follow these steps:

- Localization of spheres
- Cellular localization
- Proof of the existence

Localizing spheres

Let $S = \mathbb{Z} \setminus \{p_1\mathbb{Z} \cup \dots \cup p_i\mathbb{Z} \cup \dots\}$ be the set of integers coprime to all p_i . Denote by k_i the elements in S .

The CW-complex

$$S^{-1}\mathbb{S}^n = (V_{i=0}^{\infty} \mathbb{S}_i^n) \cup_h (V_{j=1}^{\infty} \mathbb{D}_j^{n+1})$$

(for $n \geq 1$) is the localization of $\mathbb{S}^n$ with respect to S . The attaching map h is defined as follows: the disk $\mathbb{D}_j^{n+1}$ is attached to the one-point union of spheres by the map

$$\partial \mathbb{D}_j^{n+1} = \mathbb{S}^n \rightarrow \mathbb{S}_{j-1}^n \vee \mathbb{S}_j^n$$

representing $[\mathbb{S}^n] \rightarrow [\mathbb{S}_{j-1}^n] - k_j[\mathbb{S}_j^n]$.

Localizing spheres

Hurewicz's theorem

If a space X is $(n-1)$ -connected, $n \geq 2$ then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$.

For this, note that $H_n(\mathbb{S}_{j-1}^n \vee \mathbb{S}_j^n) = \mathbb{Z} \oplus \mathbb{Z}$ and the homology group can be identified with $\pi_n(\mathbb{S}_{j-1}^n \vee \mathbb{S}_j^n)$ via the Hurewicz's theorem.

Localizing spheres

In order to prove that $i : \mathbb{S}^n \hookrightarrow S^{-1}\mathbb{S}^n$ is an S -localization, we can use the theorem presented before:

Localization (necessary and sufficient conditions)

Let Y be a simply-connected space. Y is S -local if and only if $H_*(Y, y_0; \mathbb{Z})$ is S -local. A continuous map $\phi : X \rightarrow Y$ between simply-connected spaces is an S -localization if and only if Y is S -local and $H_*(\phi; S^{-1}\mathbb{Z})$ is an isomorphism.

It is basically suffices to prove that

$$H_i(S^{-1}\mathbb{S}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ S^{-1}\mathbb{Z}, & i = n \\ 0, & \text{otherwise} \end{cases}$$

with $1 \in H_i(S^{-1}\mathbb{S}^n, \mathbb{Z})$ being represented by $[\mathbb{S}_0^n]$

Localization of the disk $(n+1)$ -disk

The localization of the $\mathbb{D}^{n+1}$ with respect to S is given by

$$S^{-1}\mathbb{D}^{n+1} = (S^{-1}\mathbb{S}^n) \times I / (S^{-1}\mathbb{S}^n) \times \{0\}$$

That is, $S^{-1}\mathbb{D}^{n+1}$ is the cone over $S^{-1}\mathbb{S}^n$, and as cone, it is contractible. Note that $S^{-1}\mathbb{S}^n$ is an S^{-1} CW-subcomplex of $S^{-1}\mathbb{D}^{n+1}$ (it satisfies $\partial S^{-1}\mathbb{D}^{n+1} = S^{-1}\mathbb{S}^n$).

Cellular localization

Definition: relative $S^{-1}\text{CW-complex}$

A relative $S^{-1}\text{CW-complex}$ is a pair of topological spaces (X, A) together with a filtration $X = \bigcup_{n=-1}^{\infty} X((n))$ of X by closed subspaces $X((n))$ such that

- $X((1)) = X((0)) = X((-1)) = A$ and A is simply-connected
- for $n \geq 1$ the space $X((n+1))$ is presented as

$$X((n+1)) = X((n)) \cup_{f_n} \left(\bigsqcup_{\alpha} S^{-1}\mathbb{D}_{\alpha}^{n+1} \right)$$

with $f_n : \bigsqcup_{\alpha} S^{-1}\mathbb{S}^n \rightarrow X((n))$ a cellular map.

- X carries the weak topology

Cellular localization

The $S^{-1}\mathbb{D}^{n+1}$ are the S -local $(n+1)$ -cells. If $A = \{pt\}$, this defines an S^{-1} CW-complex.

Examples of relative CW-complexes

- $(S^{-1}\mathbb{S}^n, \mathbb{S}^n)$
- $(S^{-1}\mathbb{D}^{n+1}, \mathbb{D}^{n+1})$
- $(S^{-1}\mathbb{D}^{n+1}, S^{-1}\mathbb{S}^n)$

Cellular localization

S-locality

Let (X, A) be a relative $S^{-1}\text{CW}$ -complex over an $S^{-1}\text{CW}$ -complex A . Then, X is S -local.

Cellular localization

The S -localization

Suppose (X, A) is a relative CW-complex with $X_1 = X_0 = A$ with A a simply-connected S -local space. Then, there is an S -localization $S^{-1}X$ of X

$$\phi : (X, A) \rightarrow (S^{-1}X, A)$$

with the properties

- The pair $(S^{-1}X, A)$ is a relative S^{-1} CW-complex
- The S -local n -cells of $(S^{-1}X, A)$ are in bijection with the n -cells of (X, A) for $n \geq 2$.
- The morphism ϕ is the inclusion of a sub-relative CW-complex

The construction of $S^{-1}X$

Let $\phi : Y \rightarrow X$ be a CW-model of X , i.e., ϕ is a weak homotopy equivalence and Y a CW-complex. By cellular approximation, we may assume that $Y_0 = Y_1 = \{pt\}$, since X is simply-connected. Let $\mu : Y \rightarrow S^{-1}Y$ be a cellular S -localization of Y .

The construction of $S^{-1}X$

We construct the localization of X as

$$S^{-1}X = X \cup_{\phi} (Y \times I) \cup_{\mu} S^{-1}Y$$

where $(y, 0)$ is identified with $\phi(y)$ and $(y, 1)$ is identified with $\mu(y)$. Then $(S^{-1}X, X)$ is a relative CW-complex without 0-cells nor 1-cells. Using this and the fact that X is simply-connected, we can see that $S^{-1}X$ is simply-connected.

The construction of $S^{-1}X$

In order to prove that $S^{-1}X$ is the localization of X , it suffices to show that the inclusion $X \hookrightarrow S^{-1}X$ induces an isomorphism in homology with coefficients in $S^{-1}\mathbb{Z}$.

Thank you for your attention



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